APY and Annuities

Finite Math

20 February 2017

Quiz

If some amount of money is deposited into a savings account with interest compounded biweekly, how many times is it compounded after 4 years?

Tutoring

The HAC has two tutors for Finite Math who are available by appointment (unless demand shows them otherwise):

- Lia Clark: liaclark@go.rmc.edu
- Jordan McCall: jordanmccall@go.rmc.edu

Compound Interest

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Example

The Russell Index tracks the average performance of various groups of stocks. On average, a $\$10,000$ *investment in mid-cap growth funds over a* 10*-year period would have grown to* $\$63,000$*. What annual nominal rate would produce the same growth if interest were compounded (a) annually, (b) continuously? Express answers as a percentage, rounded to three decimal places.*

Now You Try It!

Example

A promissory note will pay $\$50,000$ *at maturity* 6 *years from now. If you pay* $\$28,000$ *for the note now, what rate would you earn if interest were compounded (a) quarterly, (b) continuously?*

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Solution

(a) 9.78%

(b) 9.66%

Annual Percentage Yield

Suppose we are looking at various certificates of deposit (CDs) from different banks and we've come across the following ones

Bank	Rate	Compounded
Advanta	4.93%	monthly
DeepGreen	4.95%	daily
Charter One	4.97%	quarterly
Liberty	4.94%	continuously

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Charter One	4.97%	quarterly
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How can we tell which has the largest return on our investment?

Annual Percentage Yield

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Bank	**Return**
Advanta	$1,050.43
DeepGreen	$1,050.74
Charter One	$1,050.63
Liberty	$1,050.64

Annual Percentage Yield

It's best to come up with a standardized number, which we call the *Annual Percentage Yield*. What this number does is tell you how much your investment will grow by at the end of 1 year. In a sense, it is the *effective interest rate*.

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$$\begin{array}{c}\text{amount at}\\ \text{simple interest}\\ \text{after 1 year}\end{array} = \begin{array}{c}\text{amount at}\\ \text{compound interest}\\ \text{after 1 year}\end{array}$$

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$$\begin{matrix}\text{amount at} \\ \text{simple interest} \\ \text{after 1 year}\end{matrix} = \begin{matrix}\text{amount at} \\ \text{compound interest} \\ \text{after 1 year}\end{matrix}$$

Solving for the simple interest rate on the left will tell us, effectively, how much interest is made in a year.

Derivation

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\end{aligned}
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APY

Definition (Annual Percentage Yield)

If a principal is invested at the annual (nominal) rate r compounded m times a year, then the annual percentage yield is

$$APY = \left(1 + \frac{r}{m}\right)^m - 1$$

If a principal is invested at the annual (nominal) rate r compounded continuously, then the annual percentage yield is

$$APY = e^r - 1$$

APY

Example

Southern Pacific Bank offered a 1*-year CD that paid* 4.8% *compounded daily and Washington Savings Bank offered one that paid* 4.85% *compounded quarterly. Find the APY for each CD. Which has a higher return?*

Now You Try It!

Example

An online bank listed a 1*-year CD that earns* 1.25% *compounded monthly. Find the APY as a percentage, rounded to three decimal places.*

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Example

An online bank listed a 1-year CD that earns 1.25% compounded monthly. Find the APY as a percentage, rounded to three decimal places.

Solution

1.257%

Annuities

At this point, we have only discussed investments where there was one initial deposit and a final payoff. But what if you make regular equal payments into an account?

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At this point, we have only discussed investments where there was one initial deposit and a final payoff. But what if you make regular equal payments into an account? An *annuity* is a sequence of equal periodic payments. If payments are made at the end of each time interval, then the annuity if called an *ordinary annuity*. Our goal will be to find the future value of an annuity.

Future Value of an Annuity

Example

Suppose you decide to deposit \$100 *every* 6 *months into a savings account which pays* 6% *compounded semiannually. If you make* 6 *deposits, one at the end of each interest payment period over the course of* 3 *years, how much money will be in the account after the last deposit is made?*

Solution

We can visualize the value of each of those $100 deposits in a table.

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Deposit	Term	# of times Compounded	Future Value
$100	1	5	

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\$100	2		

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\$100	1	5	$\$100\left(1+\frac{0.06}{2}\right)^5 = \$100(1.03)^5$
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\$100	5	1	$\$100\left(1+\frac{0.06}{2}\right)^1 = \$100(1.03)$
\$100	6	0	$\$100\left(1+\frac{0.06}{2}\right)^0 = \$100$

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So adding up the future values of all these will give us the amount of money in the account

$$\begin{aligned} B &= \$100(1.03)^5 + \$100(1.03)^4 + \$100(1.03)^3 \\ &\quad + \$100(1.03)^2 + \$100(1.03) + \$100 = \$646.84 \end{aligned}$$